Lecture 39 (The last one... for now)

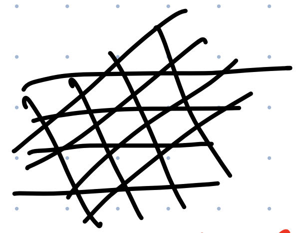
Refs
Kapovich–Leeb Disc Isom Grps of Symm Sp.
Kapovich–Leeb–Porti Dynam on Flag Mfld: domains..

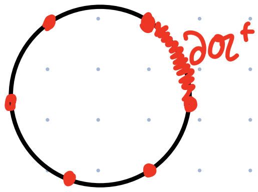
Dimension count: $G = SL_3\mathbb{R}$ (8) $K = SO(3)$ (3)

$G/K \cong \mathbb{R}^5$ $\partial_{vis}(G/K) \cong S^4$

But G/P is either $\underbrace{Flag(\mathbb{R}^3)}_{3}$ or $\underbrace{\mathbb{RP}^2}_{2}$.

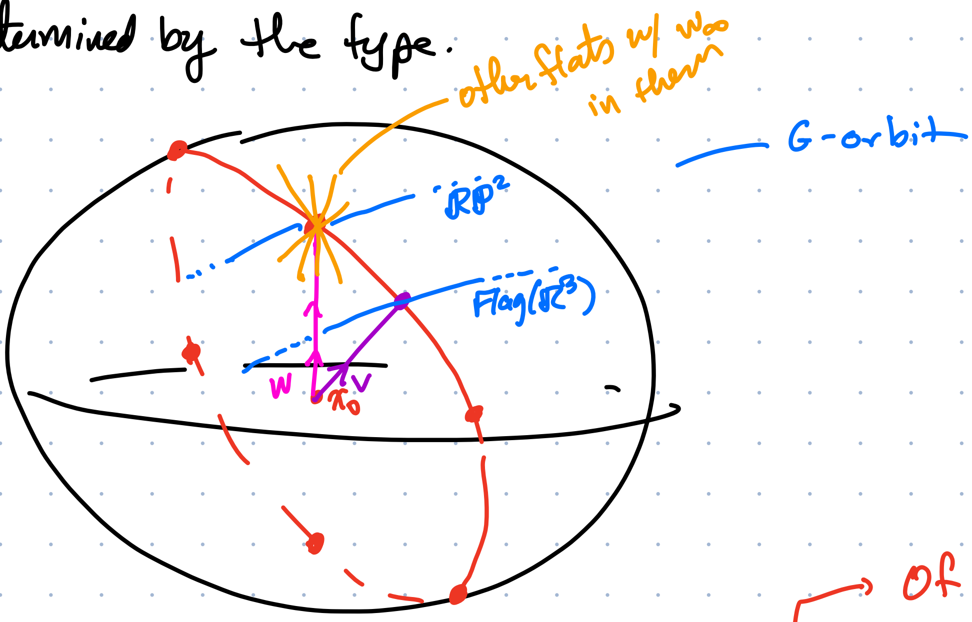
How are these put together?

$\mathfrak{a} =$ 

$\partial_\infty \mathfrak{a} =$ 

spherical coxeter complex. $(Y, W) \rightsquigarrow$ simplicial cplx homeo to S^k + simp action of coxeter grp W.
($\|v\| = 1$ in $\mathfrak{a}$ w/ W action)

Every geodesic lies in a flat. Its orbit is a G/P where P is determined by the type.



This gives $\partial_{vis} G/K$ the structure of a <u>spherical</u> <u>building</u>: of type $(\partial_\infty\mathfrak{a}, W)$

A space + a collection of <u>subspaces</u> called <u>apartments</u>.

Each apartment should have a simplicial complex structure iso to spherical coxeter cplx (fixed).

Intersections of apartments should be subcomplexes.

Closed top simplices in apartments are <u>chambers</u>. $\cong \overline{\mathfrak{a}}^+_S$

Here, apartments are unions $\partial_\infty(\text{flat in } G/K) \cong \partial_\infty \mathfrak{a}$.

Ex. $SO(1,n)$ rank 1. $\partial_\infty G/K \cong \mathbb{R}^n$ $\mathfrak{a} \cong \mathbb{R}$ ($\mathfrak{m}$ = huge)

$W \cong \mathbb{Z}/2$ $\partial_\infty \mathfrak{a} = S^0$ = two points Apartment = any pair of distinct pts in S^{n-1}

P = Stabilizer of a null line like $(1,1,0,\dots,0)$

$G/P \cong S^{n-1}$.

$\partial_{vis} G/K$ has one G-orbit.

Apartment coming from flat containing $p \in G/K$: points in S^{n-1} antipodal from p.

Each G-orbit meets each chamber in one point.

<u>Dynamics on G/K, G/P.</u>

G is the isometry group of G/K, so classifying isometries is basically describing conjugacy classes in G (or some coarser equivalence notion).

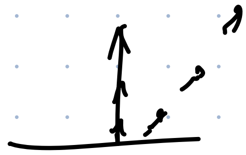
(Note that "dynamics" of a single isom isn't very interesting in that:

No attracting or repelling fixed pts.

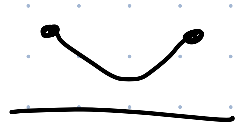
$f^n(x)$ converges iff constant. Never mixing)

<u>Hyperbolic isom in $\mathbb{H}^2 \cong \mathbb{H}$, $PSL_2\mathbb{R}$</u>

Elliptic: has a fixed pt ($\Rightarrow$ unique) $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

Hyperbolic: has invt geodesic  $\begin{pmatrix} \lambda & \\ & 1/\lambda \end{pmatrix}$ $\lambda \neq 1$

Parab.: has a fixed pt in ∂_{vis} $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Let $g \in \text{Isom}$. $d_g : X \to \mathbb{R}^{\geq 0}$ $d_g(x) = d(x, gx)$ 

Nonpos curvature $\Rightarrow$ this fn is convex along geod.

Ell: $\inf d_g = 0$ attained

Parab: $\inf d_g = 0$ not attained

Hyp: $\inf d_g > 0 \Rightarrow$ attained.

Now $X = G/K$

transvection

$\inf d_g > 0$ & realized: Hyp/axial $\quad \exists$ invt geodesic (axis)
not unique
union of them = min set (d_g)

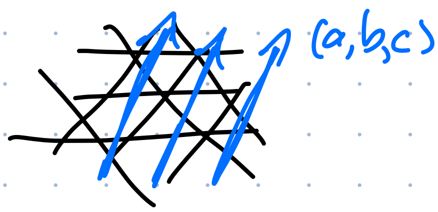
$\inf d_g > 0$ & not realized: Mixed

$\inf d_g = 0$ & realized: Elliptic $\quad$ Has fixed pt.
Union of fixed pts is tot geod & symmetric.

$\inf d_g = 0$ & not realized: Parabolic

unipotent

Ex. $SL_3\mathbb{R}$. $\begin{pmatrix} e^a & & \\ & e^b & \\ & & e^c \end{pmatrix}$ $a+b+c=0$. Translates in A by (a,b,c)
Axial



$\begin{pmatrix} e^s & 1 & \\ & e^s & \\ & & e^{-2s} \end{pmatrix}$ $\inf d_g = \inf d_{g'}$ where $g' = \begin{pmatrix} e^s & & \\ & e^s & \\ & & e^{-2s} \end{pmatrix}$

$K - \{e\}$: elliptic

$\begin{pmatrix} 1 & t & \\ & 1 & t \\ & & 1 \end{pmatrix}$ parabolic & unipotent

$$\begin{pmatrix} 1 & t & & \\ & 1 & & \\ & & \cos\theta & \sin\theta \\ & & -\sin\theta & \cos\theta \end{pmatrix}$$ parabolic in $SL_4\mathbb{R}$ but not unipotent

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Example of $G\cdot[\gamma]$ meeting each chamber:

$\gamma(t) = \exp(tv)$ $v \in \mathfrak{a}$ diagonal real.

Realize $w \in W$ as equiv class of elt of K.
Then $k_w \cdot [\gamma] = [k_w \cdot \gamma] = [\exp[t\,(k_w v)]]$
$\quad$ in a different chamber.

Convergence property.

In $PSL_2\mathbb{C} = G$ acting on $\mathbb{CP}^1 = G/B$:

let $f \in G$. Either $\{f^n\}_{n\in\mathbb{Z}}$ lies in a compact set ($\subset K$ for K cpt)

OR $\exists$ $a, b \in \mathbb{CP}^1$ s.t. on $\mathbb{CP}^1 - \{a\}$, $f^n \xrightarrow[\text{loc unif}]{}$ const fn b as $n \to +\infty$

Moreover Same for $g_n \to \infty$ in G as long as we can choose subseq.

G acts on $\mathbb{CP}^1$ as a convergence group.

same for cells C_w^a

Higher rank version: X_w Schubert cell "center x_0" $X_w^a = g X_w$ $gx_0 = a$.

If $g_n \to \infty$ and $\vec{d}(x_0, g_n \cdot x_0) \to \infty$ as does its distance from $\partial\overline{\mathfrak{a}^+}$
then $\exists$ $a, b \in G/B$ and subseq g_{n_k} s.t.

On $C^a_{w_0}$, $g_{n_k} \xrightarrow{\text{loc unif}}$ const map b